Problem 1

Let $f(x) = Ae^{-2x}$ for $0 < x < 2$ ($f(x) = 0$ for any other value of x) be a p.d.f.
For what value of A is $f(x)$ a true density function?
Using the value of A calculated above, what is the probability $P(-2 \leq x \leq 2)$?

Solution 1

For f to de a p.d.f. we must have 2 conditions:
$f(x) \geq 0$ for all x and $\int_{-\infty}^{\infty} f(x)dx = 1$.

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{2} Ae^{-2x}dx = A(\frac{-1}{2}e^{-2x}]_0^2 = \frac{-1}{2}A(e^{-4} - e^0) = \frac{1}{2}A(1 - e^{-4})$$

So since $1 = \int_{-\infty}^{\infty} f(x)dx = \frac{1}{2}A(1 - e^{-4})$ we get that $A = \frac{2}{(1-e^{-4})}$.
We don't really need the value of A calculated above to notice that $f(x) > 0$ only for $0 < x < 2$ so $P(0 \leq x \leq 2) = 1$ and hence $P(-2 \leq x \leq 2) = 1$ too.

Problem 2

Suppose that X is uniformly distributed on the interval $(-2,3)$. Let $Y = X^2$.
Find the density function of Y.
Find the distribution function of Y.

Solution 2

Since X is uniformly distributed, its p.d.f. should look like $f_X(x) = \begin{cases} A & x \in (-2,3) \\ 0 & otherwise \end{cases}$

Since $1 = \int_{-\infty}^{\infty} f_X(x)dx = \int_{-2}^{3} Adx = Ax]_{-2}^{3} = A(3-(-2)) = 5A$, we get that $A = \frac{1}{5}$. So the p.d.f. for X is $f_X(x) = \begin{cases} \frac{1}{5} & x \in (-2,3) \\ 0 & otherwise \end{cases}$

The c.d.f. of X is simply an integration over the p.d.f.: $F_X(x) = \int_{-\infty}^{x} f_X(u)du$. We must notice that this function has 3 distinct areas: $x < -2$, $x > 3$ and $-2 \le x \le 3$. For $x < -2$, $F_X(x) = 0$ clearly. For $x > 3$, $F_X(x) = 1$. The interesting part is for $-2 \le x \le 3$ where we get:

$$F_X(x) = \int_{-\infty}^{x} f_X(u)du = \int_{-2}^{x} \frac{1}{5}du = \frac{1}{5}u]_{-2}^{x} = \frac{1}{5}(x-(-2)) = \frac{1}{5}(x+2)$$

So our c.d.f. is $F_X(x) = \begin{cases} 0 & x < -2 \\ \frac{1}{5}(x+2) & -2 \le x \le 3 \\ 1 & x > 3 \end{cases}$

Now $Y = X^2$, so if $x \in (-2,3)$ then $y \in (0,9)$. We will split the problem into 2 areas: $0 \le y \le 4$ (i.e. where $-2 \le x \le 2$) and $4 < y \le 9$ (i.e. for $2 < x \le 3$).
Lets calculate the c.d.f. for $y \in (0,4)$.

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) \\ &= P(X \le \sqrt{y}) - P(X \le -\sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}). \end{aligned}$$

Since $y \in (0,4)$ we get that $x = \sqrt{y} \in (0,2)$ and $x = -\sqrt{y} \in (-2,0)$ so $x \in (-2,2)$ and we can plug it into the c.d.f. from above to get

$$F_X(\sqrt{y}) - F_X(-\sqrt{y}) = \frac{1}{5}(\sqrt{y}+2) - \frac{1}{5}(-\sqrt{y}+2) = \frac{2\sqrt{y}}{5}$$

For $y \in (4,9)$ the situation is simpler, since we definitly know that $x \in (2,3)$.

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(X \le \sqrt{y}) = P(X \le \sqrt{y}) = F_X(\sqrt{y}) = \frac{1}{5}(\sqrt{y}+2)$$

We combine our results and get Y's c.d.f. to be: $F_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{2\sqrt{y}}{5} & 0 \le y \le 4 \\ \frac{1}{5}(\sqrt{y}+2) & 4 < y \le 9 \\ 1 & y > 9 \end{cases}$

For the p.d.f we can derive the c.d.f. and get $f_Y(y) = \begin{cases} \frac{1}{5\sqrt{y}} & 0 \le y \le 4 \\ \frac{1}{10\sqrt{y}} & 4 < y \le 9 \\ 0 & otherwise \end{cases}$

Problem 3

Suppose that X and Y are independent random variables, each uniformly distributed on the interval $(0, 1)$. Let $V = X - Y$ and let $W = X + Y$.
Find the distribution function of V.
Find the distribution function of W.

Solution 3

We have already seen that the p.d.f. and c.d.f. of a uniformly distributed random variables such as X and Y should be:
$f(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & otherwise \end{cases}$ and $F(x) = \begin{cases} 0 & x < 0 \\ x & x \in (0,1) \\ 0 & x > 1 \end{cases}$
For any value v of $V = X - Y$ there are many options of what values X and Y can get. We can be more percise and determine that for any value v, if $X = x$ then Y must equal $(x - v)$ so that $V = X - Y = x - (x - v) = v$.
To get the p.d.f. of V at point v we will then integrate over all possible values of x. The p.d.f. can be written as $f_V(v) = \int_{-\infty}^{\infty} f_X(x) f_Y(x - v)dx$
Lets look at the p.d.f. of V when v is positive $(0 \leq v \leq 1)$. Notice that while both x and y are in $(0, 1)$, v can get values between -1 and 1.
Our multiplicants are non zero when $0 \leq x \leq 1$ and $0 \leq x - v \leq 1$ which in this case $(0 \leq v \leq 1)$means $v \leq x \leq 1 + v$. So the integration limits will be $v \leq x \leq 1$:

$$f_V(v) = \int_v^1 f_X(x) f_Y(x - v)dx = \int_v^1 1 * 1dx = \int_v^1 1dx = x]_v^1 = 1 - v$$

When $-1 \leq v \leq 0$, our integration limits change into $0 \leq x \leq 1 + v$ and we get:

$$f_V(v) = \int_0^{1+v} f_X(x) f_Y(x - v)dx = \int_0^{1+v} 1 * 1dx = \int_0^{1+v} 1dx = x]_0^{1+v} = v + 1$$

We combine our results and get the p.d.f. $f_V(v) = \begin{cases} v + 1 & -1 \leq v \leq 0 \\ 1 - v & 0 \leq v \leq 1 \\ 0 & otherwise \end{cases}$
To get the c.d.f. we must integrate the p.d.f. to get:
For $v \in (-1, 0)$,

$$F_V(v) = \int_{-1}^{v} f_V(u)du = \int_{-1}^{v} (u+1)du = \frac{u^2}{2} + u]_{-1}^{v} = \frac{v^2}{2} + v - (\frac{1}{2} - 1) = \frac{v^2}{2} + v + \frac{1}{2}$$

For $v \in (0, 1)$,

$$F_V(v) = \int_{-1}^{v} f_V(u)du = \int_{-1}^{0} (u+1)du + \int_{0}^{v} (1-u)du = \frac{1}{2} + (u - \frac{u^2}{2})]_0^v = \frac{1}{2} + v - \frac{v^2}{2}$$

And finally $F_V(v) = \begin{cases} 0 & v < -1 \\ \frac{v^2}{2} + v + \frac{1}{2} & -1 \leq v \leq 0 \\ \frac{1}{2} + v - \frac{v^2}{2} & 0 \leq v \leq 1 \\ 1 & 1 < v \end{cases}$

$W = X + Y$ is very similar to calculate.
For any value w of $W = X + Y$ there are many options of what values X and Y can get. For any value w, if $X = x$ then Y must equal $(w - x)$ so that $W = X + Y = x + (w - x) = w$.
To get the p.d.f. of W at point w we will then integrate over all possible values of x. The p.d.f. can be written as $f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w - x)dx$
Notice that this time since both x and y are in $(0, 1)$, w can get values between 0 and 2.
Our multiplicants are non zero when $0 \le x \le 1$ and $0 \le w - x \le 1$ which in our case means $w - 1 \le x \le w$.
Again it will be comfortable splitting the function into 2 areas namely $w \in (0, 1)$ and $w \in (1, 2)$ so that our integration includes non-zero values only.
For $w \in (0, 1)$ the integration limits will be $0 \le x \le w$:

$$f_W(w) = \int_0^w f_X(x) f_Y(w - x)dx = \int_0^w 1 * 1dx = \int_0^w 1dx = x]_0^w = w$$

For $w \in (1, 2)$ the integration limits will be $w - 1 \le x \le 1$ and we get:

$$f_W(w) = \int_{w-1}^1 f_X(x) f_Y(w - x)dx = \int_{w-1}^1 1dx = x]_{w-1}^1 = 2 - w$$

We combine our results and get the p.d.f. $f_W(w) = \begin{cases} w & 0 \le w \le 1 \\ 2 - w & 1 \le w \le 2 \\ 0 & otherwise \end{cases}$
To get the c.d.f. we must integrate the p.d.f. to get:
For $w \in (0, 1)$,

$$F_W(w) = \int_0^w f_W(u)du = \int_0^w udu = \frac{u^2}{2}]_0^w = \frac{w^2}{2} - 0 = \frac{w^2}{2}$$

For $w \in (1, 2)$,

$$F_W(w) = \int_0^1 f_W(u)du + \int_1^w f_W(u)du = \frac{1}{2} + \int_1^w (2 - u)du =$$

$$\frac{1}{2} + (2u - \frac{u^2}{2})]_1^w = \frac{1}{2} + (2w - \frac{w^2}{2}) - (2 - \frac{1}{2}) = 2w - \frac{w^2}{2} - 1$$

Finally we get $F_W(w) = \begin{cases} 0 & w < 0 \\ \frac{w^2}{2} & 0 \le w \le 1 \\ 2w - \frac{w^2}{2} - 1 & 1 \le w \le 2 \\ 1 & 2 < w \end{cases}$

$$Note: \ There\ are\ many\ other\ ways\ of\ finding\ these\ functions.$$

Problem 4

Suppose that X and Y are independent random variables and each is of exponential distribution with mean $\frac{1}{3}$, i.e. $f(x) = 3e^{-3x}$ and $f(y) = 3e^{-3y}$. Let $V = X + Y$, let $W = min(X, Y$ and let $Z = max(X, Y)$.
Find the distribution function of V.
Find the distribution function of W.
Find the distribution function of Z.

Solution 4

For $V = X + Y$ we will follow the same principle as before:
For any value v of $V = X + Y$ there are many options of what values X and Y can get. For any value v, if $X = x$ then Y must equal $(v - x)$ so that $W = X + Y = x + (v - x) = w$.
To get the p.d.f. of V at point v we will then integrate over all possible values of x. The p.d.f. can be written as $f_V(v) = \int_{-\infty}^{\infty} f_X(x)f_Y(v-x)dx$
Notice that this time since both x and y are in $(0, \infty)$, $v = x + y$ also has the same range.
Our multiplicants are non zero when $0 \leq x \leq \infty$ and $0 \leq v - x \leq \infty$ which in our case means $-\infty \leq x \leq v$. Combining these 2 conditions we get that the integration limits should be $0 \leq x \leq v$:

$$f_V(v) = \int_0^v f_X(x)f_Y(v-x)dx = \int_0^v 3e^{-3x} * 3e^{-3(v-x)}dx = \int_0^v 9e^{-3v}dx =$$

$$9e^{-3v}\int_0^v dx = 9e^{-3v}x]_0^v = 9e^{-3v}(v-0) = 9ve^{-3v}$$

To get the c.d.f. we must integrate the p.d.f. to get:

$$F_V(v) = \int_0^v f_V(u)du = \int_0^v 9ue^{-3u}du =$$

To evaluate $\int_0^v 9ue^{-3u}du$ we will use integration by parts ($\int xdy = xy - \int ydx$).
Set $x = 9u$ and $dy = e^{-3u}du$. So $dx = 9du$ and $y = \int e^{-3u}du = \frac{-1}{3}e^{-3u}$
Now integrate

$$\int_0^v 9ue^{-3u}du = (9u * \frac{-1}{3}e^{-3u})]_0^v - \int_0^v \frac{-1}{3}e^{-3u} * 9du = -3ue^{-3u}]_0^v - \int_0^v 3e^{-3v}dv$$

$$= (-3ve^{-3v} - 0) - (-e^{-3u})]_0^v = -3ve^{-3v} - (-e^{-3v} - (-1)) = 1 - (3v+1)e^{-3v}$$

So $F_V(v) = \begin{cases} 0 & v < 0 \\ 1 - (3v+1)e^{-3v} & v > 0 \end{cases}$

Lets start with $Z = max(X, Y)$. What does the fact that Z is $max(X, Y)$ mean?
It means that if $Z \leq z$ for some z, then both X and Y are smaller than z.
In probability notation we can say that $P(Z \leq z) = P(X \leq z \; and \; Y \leq z)$.
Since X and Y are independant we can write this as $P(Z \leq z) = P(X \leq z) \; P(Y \leq z)$.
By definition $P(X \leq z) = F_X(z)$ and $P(Y \leq z) = F_Y(z)$.
So all we need to do is find the c.d.f. of the exponential function.

$$F(x) = \int_0^x 3e^{-3u} = -e^{-3u}]_0^x = (-e^{-3v} - 1) = 1 - e^{-3v}$$

So the c.d.f. is (for $z \geq 0$):

$$F_Z(z) = P(Z \leq z) = P(X \leq z) * P(Y \leq z) = F_X(z) \; F_Y(z) = (1 - e^{-3z})^2 =$$
$$1 - 2e^{-3z} + e^{-6z}$$

With $W = min(X, Y)$ we go through a similar process.
Here, when $W \leq w$, it means that the minimum of X and Y is less than w. We can rephrase it as when the minimum is greater than w ($W > w$), then both X and Y must be also greater than w. To put this in probablistic notation: $P(W > w) = P(X > w \; and \; Y > w) = P(X > w) \; P(Y > w)$ since X and Y are independant.
We know that $P(W \leq w) = F_W(w)$, but whar about $P(W > w)$?
$P(W > w)$ is the complement of $P(W \leq w)$, so $P(W > w) = 1 - P(W \leq w)$
Now we are ready to calculate our c.d.f.

$$F_W(w) = P(W \leq w) = 1 - P(w > w) = 1 - P(X > w) \; P(Y > w)$$

$P(X > w) = 1 - P(X \leq w) = 1 - F_X(w)$ for the same reasons as before and we finally get:

$$F_W(w) = 1 - P(X > w) \; P(Y > w) = 1 - (1 - P(X \leq w))(1 - P(Y \leq w)) =$$
$$1 - (1 - F_X(w))(1 - F_Y(w)) = 1 - (1 - F_X(w) - F_Y(w) + F_X(w)F_Y(w)) =$$
$$1 - (1 - (1 - e^{-3w}) - (1 - e^{-3w}) + (1 - e^{-3w})^2) =$$
$$1 - (1 - 1 + e^{-3w} - 1 + e^{-3w} + 1 - 2e^{-3w} + e^{-6w}) = 1 - e^{-6w}$$

Here too, the domain is $(0, \infty)$.

$$Note: \; There \; are \; many \; other \; ways \; of \; finding \; these \; functions.$$